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Quantization scheme based on the extension of phase space

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Abstract. A new quantization scheme is proposed. The similarity of geometric and covariant quantization of a certain class of constrained dynamical systems is shown. A method to calculate anomalies is developed.

1. Introduction

Quantization of dynamical systems is one of the central problems of mathematical physics. At the same time the procedure of constructing a quantum theory corresponding to a given classical system is not unique. Various different approaches have been considered (see e.g. [1–4] and references therein).

A quantization scheme which we discuss in this note is, in a certain sense, opposite to the scheme of the non-covariant quantization [1]. For a given initial Hamiltonian system we construct a (classically) equivalent extended system with constraints. Then the extended system is quantized by the covariant method||.

Of course, there are many different possibilities to extend the initial dynamical system. We propose a 'minimal' scheme of extension, which, at the same time, is quite general. It can be used for finite and for infinite dimensional cases. Although our quantization method includes several steps, its main advantage is the simple form of the physical operators in the extended space. In particular, this simplifies the operator ordering problem and the consideration of anomalies.

The covariant quantization of the constructed extended systems turns out to be very similar to the geometric quantization [2] of the initial systems. In our quantization scheme, the so called prequantization operators of the geometric quantization arise as well, and the restriction (by constraints) of the extended Hilbert space to the physical subspace is analogous to the choice of polarization in the geometric quantization [2]. Thus, we have the following scheme:

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|| For simplicity, we assume that the initial dynamical system has no constraints.

$$\begin{tikzcd}
\boxed{\text{Extended Systems with constraints}} \arrow[r, "Cov.Quant."] & \boxed{\text{Quantization in all coordinates}} \arrow[r] \arrow[d, leftrightarrow] & \boxed{\text{Restriction of quantum space}} \arrow[d, leftrightarrow] \\
\boxed{\text{Hamiltonian Systems}} \arrow[u] \arrow[r, "Geom.Quant."] & \boxed{\text{pre-quantization}} \arrow[r] & \boxed{\text{Choice of polarization}}
\end{tikzcd}$$

In the following we shall briefly consider the quantization scheme based on the extension of phase space by second class constraints†.

The current paper is organized as follows. In section 2 our extension scheme is described. In section 3 we consider the covariant quantization of the obtained extended systems. As an application, we give a method to calculate the anomalies in the commutation relations of the physical operators. Section 4 contains concluding remarks.

2. The scheme of extension of phase space

Let us consider the symplectic manifold γ with local coordinates $\xi^k\,(k = 1, 2, \ldots, 2N)$ and symplectic 2-form ω

$$\omega = \tfrac{1}{2}\omega_{kl}(\xi)\mathrm{d}\xi^k \wedge \mathrm{d}\xi^l .$$

The anti-symmetric matrix $\omega_{kl}(x)$ defines the Hamiltonian structure in γ (see e.g. [6]). Indeed:

(i) For any smooth function $H(x)$ in γ one can construct the Hamiltonian vector field

$$F_H = F_H^k \partial_k \tag{2.1}$$

with components

$$F_H^k = \omega^{kl}(\xi)\partial_l H(\xi)$$

and correspondingly obtain the equations of motion

$$\dot{\xi}^k = F_H^k(\xi). \tag{2.2}$$

(ii) For any smooth functions G and H on γ one can define the Poisson brackets (PB) as

$$\{G, H\}_\gamma \boxminus \omega^{kl}\partial_l G \partial_k H \tag{2.3}$$

and thereby introduce the Lie algebra structure.

† More details and the scheme of quantization with the extension of phase space by first class constraints are considered in [5].

Here $\omega^{kl}(\xi)$ is the inverse matrix of $\omega_{kl}(\xi)$

$$\omega_{kl}\omega^{lj} = \delta^l_k \qquad \partial_k \equiv \frac{\partial}{\partial \xi^k}.$$

For an exact 2-form ω, $\omega = \mathrm{d}\theta$, the matrix ω_{kl} can be expressed in terms of the components of the 1-form θ

$$\omega_{kl} = \partial_k\theta_l - \partial_l\theta_k \tag{2.4}$$

where $\theta = \theta_k \mathrm{d}\xi^k$. In this case the Hamilton equations (2.2) can be obtained by the variation of the action

$$S = \int [\theta_k(\xi)\dot{\xi}^k - H(\xi)]\,\mathrm{d}t. \tag{2.5}$$

The standard method of quantization of a system with the action (2.5) consists of choosing Darboux coordinates and then performing canonical quantization [7]. This method is problematic in the case when the canonical variables are defined in a bounded region, e.g. when the phase space is compact [8]. The construction of Darboux coordinates in an explicit form, generally speaking, is also a non-trivial problem.

In order to avoid these difficulties we propose the following quantization scheme†.

Let us consider the integrand in (2.5) as the Lagrangian in the Lagrange formalism

$$L(\xi, \dot{\xi}) = \theta_k(\xi)\dot{\xi}^k - H(\xi). \tag{2.6}$$

Then, we carry out the Legendre transformation. Since the Lagrangian (2.6) is singular

$$\frac{\partial^2 L}{\partial\dot{\xi}^k \partial\dot{\xi}^l} = 0$$

one gets in the Hamilton description a constrained dynamical system [1].

Introducing the momentum variables

$$P_k = \frac{\partial L}{\partial \dot{\xi}^k}$$

we get the extended phase space Γ with the coordinates ξ^k and P_k. From the Lagrangian (2.6) we obtain the constraints

$$\phi_k = P_k - \theta_k(\xi) \tag{2.7}$$

and the canonical Hamiltonian

$$H_c \equiv \frac{\partial L}{\partial \dot{\xi}^k}\dot{\xi}^k - L = H(\xi).$$

† A similar point of view on the quantization of systems with action (2.5) was presented in [9] where the Dirac brackets formalism was used in order to remove the unphysical degrees of freedom. In contrast to [9], in what follows we will consider a quantization scheme based on the Gupta–Bleuler-type reduction of unphysical degrees of freedom.

Hence, we derive the action [1]

$$S = \int P_k \mathrm{d}\xi^k - [H(\xi) + \lambda^k(t)\phi_k]\mathrm{d}t \tag{2.8}$$

where λ^k are the Lagrange multipliers.

For the PB of the constraints (in Γ) one can check that

$$\{\phi_k, \phi_l\} \equiv \frac{\partial \phi_k}{\partial p_j}\frac{\partial \phi_l}{\partial \xi^j} - \frac{\partial \phi_k}{\partial \xi^j}\frac{\partial \phi_l}{\partial p_j} = \omega_{lk}. \tag{2.9}$$

Since $\det \omega_{kl} \equiv \Omega \neq 0$, the constraints ϕ_k are of second class. The Lagrange multipliers are obtained from the consistency conditions

$$\dot{\phi}_l = \{H + \lambda^k \phi_k, \phi_l\} = 0.$$

This gives

$$\lambda^k = \omega^{kl}\partial_l H.$$

The total Hamiltonian now takes the form

$$H_{\text{tot}} = H + \omega^{kl}\partial_l H \phi_k \equiv R_H(\xi, P). \tag{2.10}$$

As a result we obtain a system with the action

$$S = \int P_k \mathrm{d}\xi^k - R_H(\xi, P)\mathrm{d}t \tag{2.11}$$

and with the constraints given in (2.7). It is clear that the restriction of the system (2.11) by the constraints (2.7) gives the initial system (2.5). Thus, (2.11) together with (2.7) defines the desired extension of the initial Hamiltonian system (2.5).

Furthermore, in analogy to (2.10), for any smooth function $G(\xi)$ in γ we can construct the corresponding function $R_G(\xi, P)$ in Γ

$$G(\xi) \to R_G(\xi, P) = G(\xi) + \omega^{kl}(\xi)\partial_l G(\xi)\phi_k(\xi, P). \tag{2.12}$$

This class of functions in Γ will be called the observable functions and the manifold restricted by the constraints (2.7) will be called the physical subspace ($\Gamma_{\text{ph}} = \gamma$). Note that these functions are linear in momentum variables P_k . Using (2.3) and (2.9) one checks that

$$\{R_G, R_H\} = R_{\{G,H\}_\gamma}. \tag{2.13}$$

Therefore, the mapping (2.12) preserves the PB of observables.

Sometimes it is convenient to change from the coordinates P_k, ξ^k to the variables ϕ_k, χ^k, where

$$\chi^k \equiv R_{\xi^k} = \xi^k + \omega^{jk}(P_j - \theta_j). \tag{2.14}$$

For the PB of the new variables we have

$$\begin{aligned}\{\chi^k, \chi^l\} &= \omega^{lk} + \omega^{ij}\partial_j(\omega^{lk})\phi_i \\ \{\chi^k, \phi_l\} &= -\partial_l \omega^{jk}\phi_j\end{aligned} \tag{2.15}$$

and

$$\{R_G, \chi^k\} = \omega^{kj}\partial_j G + \omega^{il}\partial_l(\omega^{kj}\partial_j G)\phi_i \tag{2.16a}$$

$$\{R_G, \phi^k\} = -\partial_k(\omega^{jl}\partial_l G)\phi_j. \tag{2.16b}$$

For any function F in Γ, $F(\xi, P) \equiv \widetilde{F}(\chi, \phi)$, we define the restriction to the physical subspace as

$$F(\xi, P)|_\gamma = F(\xi, \theta(\xi)) = \widetilde{F}(\xi, 0).$$

Then we have for the observables

$$R_G(\xi, P)|_\gamma = G(\xi).$$

The PB (2.16) take the form

$$\{R_G, \chi^k\}\,|_\gamma = \omega^{kj}\partial_j G \qquad \{R_G, \phi^k\}\,|_\gamma = 0. \tag{2.17}$$

Thus, the physical subspace is invariant under the canonical transformations generated by the observable functions R_G. The coordinates in γ ($\chi^k|_\gamma = \xi^k$) are transformed in the same way as in the initial Hamiltonian theory (2.5) by the corresponding function $G(\xi)$.

We conclude this section with the following comment. A change of the observable functions R_G by terms of quadratic or higher order in the variables ϕ_k

$$R_G \to \widetilde{R}_G = R_G + C^{lk}\phi_l\phi_k + \cdots \tag{2.18}$$

produces the functions $\widetilde{R}_G$ with the same properties (2.17) as the initial R_G. So, for any function $G(\xi)$ there exists the class of functions (2.18) which are undistinguishable in the physical subspace. R_G is the simplest one from this class.

3. The covariant quantization of extended systems

3.1. Operators in the extended Hilbert space

If the constraints are ignored, the quantization of the system (2.11) is trivial, and it is natural to choose the coordinate representation. Then, the Hilbert space is $\mathrm{L}_2(\gamma)$ with the invariant measure

$$\langle\psi_2|\psi_1\rangle = \int \frac{\mathrm{d}^{2N}\xi\sqrt{\Omega(\xi)}}{(2\pi\hbar)^N}\psi_2^*(\xi)\psi_1(\xi).$$

The coordinate operator $\hat{\xi}^k \equiv \xi^k$ is the multiplication operator while the self-conjugate momentum operator takes the form

$$\hat{P}_k = -\mathrm{i}\hbar\partial_k - \frac{\mathrm{i}\hbar}{4\Omega}\partial_k\Omega.$$

Then, for the observable operators $\hat{R}_G$ from (2.7) and (2.12), we get

$$\hat{R}_G = G(\xi) - \theta_k(\xi)\omega^{kl}(\xi)\partial_l G(\xi) - \mathrm{i}\hbar\omega^{kl}(\xi)\partial_l G(\xi)\partial_k$$

which by using (2.1) takes the form

$$\hat{R}_G = G - \theta(F_G) - \mathrm{i}\hbar F_G. \tag{3.1a}$$

Thus, we get just the operators of prequantization [2].

Note that the operator ordering problem has been solved by the symmetric choice

$$\omega^{kl}(\xi)\partial_l G(\xi)P_k \rightarrow \tfrac{1}{2}[\hat{P}_k\omega^{kl}(\xi)\partial_l G(\xi) + \omega^{kl}(\xi)\partial_l G(\xi)\hat{P}_k].$$

One checks that the operators $\hat{R}_G$ are self-conjugate in the Hilbert space under consideration. They preserve the Lie algebra structure, (cf (2.13))

$$[\hat{R}_G, \hat{R}_H] = -\mathrm{i}\hbar\hat{R}_{\{G,H\}_\gamma}. \tag{3.2}$$

As an example, let us consider the two-dimensional flat case with $\gamma = \mathrm{R}^2$. Introduce the global Darboux coordinates q and p

$$\xi^1, \xi^2 \rightarrow q, p.$$

Then θ and ω take the form

$$\theta = p\mathrm{d}q + \mathrm{d}F(q,p) \qquad \omega_{kl} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.3}$$

with an arbitrary function $F(q,p)$.

For the observable operators (3.1) and the constraints (2.7) one gets

$$\hat{R}_G = G(q,p) + \partial_p G\hat{\phi}_q - \partial_q G\hat{\phi}_p \tag{3.1b}$$

$$\hat{\phi}_q = -\mathrm{i}\hbar\partial_q - p - \partial_q F(q,p) \qquad \hat{\phi}_p = -\mathrm{i}\hbar\partial_p - \partial_p F(q,p). \tag{3.1c}$$

Note that the operators $\hat{R}_G$ with different arbitrary functions F are related by unitary transformations

$$\hat{R}_G^{(F)} = \exp\left[\frac{\mathrm{i}}{\hbar}F(q,p)\right] \qquad \hat{R}_G^{(F=0)}\exp\left[-\frac{i}{\hbar}F(q,p)\right]. \tag{3.4}$$

For convenience we choose $F = -qp/2$. Then the operators which correspond to the coordinate functions q and p take the form (cf (2.14))

$$\hat{R}_q = \tfrac{1}{2}q + \mathrm{i}\hbar\partial_p \equiv \hat{\chi}_q \qquad \hat{R}_p = \tfrac{1}{2}p - \mathrm{i}\hbar\partial_q \equiv \hat{\chi}_p. \tag{3.5}$$

For the operators of the constraints (3.1c) we have

$$\hat{\phi}_q = -\tfrac{1}{2}p - \mathrm{i}\hbar\partial_q \qquad \hat{\phi}_p = \tfrac{1}{2}q - \mathrm{i}\hbar\partial_p. \tag{3.6}$$

Considering instead of the basis $\hat{q}, \hat{p}, \hat{P}_q, \hat{P}_p$ the basis of operators $\hat{\chi}_q, \hat{\chi}_p, \hat{\phi}_q, \hat{\phi}_p$ we arrive at the following commutation relations:

$$[\hat{\chi}_q, \hat{\chi}_p] = \mathrm{i}\hbar \qquad [\hat{\phi}_q, \hat{\phi}_p] = -\mathrm{i}\hbar \qquad [\hat{\chi}, \hat{\phi}] = 0 \tag{3.7a}$$

$$\begin{aligned} [\hat{R}_G, \hat{\phi}_q] &= \mathrm{i}\hbar(\partial^2_{qp}G\hat{\phi}_q - \partial^2_{qq}G\hat{\phi}_p) \\ [\hat{R}_G, \hat{\phi}_p] &= \mathrm{i}\hbar(\partial^2_{pp}G\hat{\phi}_q - \partial^2_{qp}G\hat{\phi}_p). \end{aligned} \tag{3.7b}$$

These are the quantum analogues of (2.15) and (2.16b).

3.2. Choice of the quantum physical subspace

Having in mind our initial system given by action (2.5), we see that the Hilbert space $L_2(\gamma)$ describes superficial degrees of freedom. We must restrict it by using the constraints. For simplicity let us consider again the two-dimensional flat case (3.3). Note that it can be easily generalized to the multi-dimensional case (see e.g. subsection 3.4).

We cannot simultaneously impose the conditions $\hat{\phi}_q|\psi\rangle = 0$ and $\hat{\phi}_p|\psi\rangle = 0$ on the state vector, since these contradicts to the commutation relations (3.7*a*). Note that the solution consistent with only one of the conditions $\hat{\phi}_q|\psi\rangle = 0$ or $\hat{\phi}_p|\psi\rangle = 0$ does not belong to the Hilbert space $L_2(R^2)$.

From quantum mechanics it is known [10] that a quantum state $|\psi\rangle$ which corresponds to the classical state $p = q = 0$ is the vacuum in the coherent state representation and is defined as a solution to the equation $(\hat{q} + i\hat{p})|\psi\rangle = 0$.

We shall do the same and choose the physical subspace as a vacuum with respect to the variables ϕ_q, ϕ_p. Thus, we introduce the 'creation' and 'annihilation' operators

$$\hat{\phi}_\alpha = \frac{(\alpha\hat{\phi}_p + i\hat{\phi}_q)}{\sqrt{2\alpha\hbar}} \qquad \hat{\phi}^*_\alpha = \frac{(\alpha\hat{\phi}_p - i\hat{\phi}_q)}{\sqrt{2\alpha\hbar}} \tag{3.8}$$

where α is a real parameter. The vectors of the physical Hilbert subspace H^α_{ph} are determined as the solutions to the equation [4]

$$\hat{\phi}_\alpha|\psi_{ph} >= 0. \tag{3.9}$$

Note that the physical subspaces H^α_{ph} for different α are unitary equivalent

$$H^\beta_{ph} = \exp\left(\frac{i}{\hbar}\kappa\hat{A}\right) H^\alpha_{ph} \tag{3.10}$$

where $\kappa = \ln(\beta/\alpha)/2$ and $\hat{A} = (\hat{\phi}_p\hat{\phi}_q + \hat{\phi}_q\hat{\phi}_p)/2$.

Using (3.6), from (3.9) we obtain the general solution of the form

$$\Psi_{ph}(q,p) = \exp\left[-\frac{1}{4\alpha\hbar}(p^2 + \alpha^2 q^2)\right] \psi\left[\frac{(\alpha q - ip)}{\sqrt{2\alpha\hbar}}\right] \tag{3.11}$$

with ψ an arbitrary function.

Now, if one introduces the complex variables

$$z = \frac{(\alpha q + ip)}{\sqrt{2\alpha\hbar}} \qquad z^* = \frac{(\alpha q - ip)}{\sqrt{2\alpha\hbar}} \tag{3.12}$$

then the solution (3.11) can be written as follows

$$\Psi_{ph}(z, z^*) = \exp(-\tfrac{1}{2}zz^*)\psi(z^*). \tag{3.13}$$

For the scalar product of the physical states one gets

$$\langle\Psi_{ph,2}|\Psi_{ph,1}\rangle = \int \frac{dz\, dz^*}{2\pi} \exp(-zz^*)\psi^*_2(z^*)\psi_1(z^*). \tag{3.14}$$

Thus, we have rederived the well known holomorphic representation [3, 11].

To further clarify this result we introduce the creation and annihilation operators $\hat{R}_z$ and $\hat{R}_{z^*}$ according to (3.5) and (3.12)

$$\hat{R}_z = \tfrac{1}{2}z + \partial_{z^*} \qquad \hat{R}_{z^*} = \tfrac{1}{2}z^* - \partial_z \tag{3.15}$$

and check that

$$\begin{aligned}\hat{R}_z[\exp(-\tfrac{1}{2}zz^*)\,\psi(z^*)] &= \exp(-\tfrac{1}{2}zz^*)\frac{\mathrm{d}}{\mathrm{d}z^*}\psi(z^*)\\ \hat{R}_{z^*}[\exp(-\tfrac{1}{2}zz^*)\,\psi(z^*)] &= \exp(-\tfrac{1}{2}zz^*)z^*\psi(z^*).\end{aligned} \tag{3.16}$$

Therefore, we have the correspondence

$$\hat{R}_z \to \hat{a} = \frac{\mathrm{d}}{\mathrm{d}z^*} \qquad \hat{R}_{z^*} \to \hat{a}^* = z^* \tag{3.17}$$

with the annihilation and creation operators, $\hat{a}$ and $\hat{a}^*$, from the holomorphic representation [3, 11].

Note that from the obtained representation, (3.1a) and (3.11), by choosing different α and $F(q,p)$, one can obtain a different unitary equivalent representation (see (3.4), (3.10)). In particular, the ordinary coordinate and momentum representations appear in the limits $\alpha \to \infty (F = 0)$ and $\alpha \to 0 (F = -pq)$, respectively.

Indeed, in the case $F = 0$, the operators $\hat{\phi}_q$ and $\hat{\phi}_p$ take the form (cf (3.6))

$$\hat{\phi}_q = -p - \mathrm{i}\hbar\partial_q \qquad \hat{\phi}_p = -\mathrm{i}\hbar\partial_p.$$

Then, due to (3.8) and (3.9) the vectors of physical subspace are the functions

$$\Psi_{\mathrm{ph}}(q,p) = \left(\frac{1}{\pi\alpha}\right)^{1/4} \exp\left(-\frac{p^2}{2\alpha}\right)\psi\left(q - \frac{\mathrm{i}}{\alpha}p\right)$$

with an arbitrary ψ. The scalar product takes the form

$$\langle\Psi_{ph,1}|\Psi_{ph,2}\rangle_\alpha = \int \mathrm{d}p\,\mathrm{d}q\left(\frac{1}{\pi\alpha}\right)^{1/2} \exp\left(-\frac{p^2}{\alpha}\right)\psi_1^*\left(q - \frac{\mathrm{i}}{\alpha}p\right)\psi_2\left(q - \frac{\mathrm{i}}{\alpha}p\right)$$

and in the limit $\alpha \to \infty$ we get

$$\langle\Psi_{ph,1}|\Psi_{ph,2}\rangle_\infty = \int \mathrm{d}q\,\psi_1^*(q)\psi_2(q)$$

where $\psi_{1,2}(q)$ are the solutions of (3.9) in the case of $\alpha \to \infty$ ($F = 0$). Similarly, it is easy to check that in the case of $\alpha = 0$ ($F = -qp$) the momentum representation will be obtained.

3.3. *Observable operators in the physical subspace*

As it was mentioned in the previous section, the classical physical subspace $\Gamma_{\rm ph}$ is invariant under the transformations generated by the class of equivalent observable functions $\widetilde{R}_G$ (see (2.17), (2.18)). In the quantum case the situation is different. In the quantum physical subspace $\mathrm{H}^\alpha_{\rm ph}$ one gets different operators for the classically equivalent functions since, in general

$$C^{lk}(\xi)\hat{\phi}_l\hat{\phi}_k + \cdots)|\Psi_{\rm ph}\rangle \neq 0.$$

Thereby, the classical equivalence is broken. On the other hand, using (3.7*b*) one can check that for the observable operators $\hat{R}_G$ the invariance condition of $\mathrm{H}^\alpha_{\rm ph}$ ($\hat{R}_G\mathrm{H}^\alpha_{\rm ph} \subset \mathrm{H}^\alpha_{\rm ph}$) restricts the class of real functions $G(q,p)$

$$G(q,p) = a_0 + a_1 q + a_2 p + a_3(p^2 + \alpha^2 q^2) \tag{3.18}$$

where $a_k\,(k = 0,1,2,3)$ are real numbers. For functions $G(q,p)$ different from the class (3.18), one has a problem of defining the corresponding operators in the quantum physical subspace. It will be shown later that the class $\widetilde{R}_G$ includes such operators.

The geometric structure of the Hilbert space implies the following definition of the operators in the physical Hilbert subspace

$$\hat{R}_{G,ph} = \hat{P}_\alpha \hat{R}_G \hat{P}_\alpha. \tag{3.19}$$

Here $\hat{P}_\alpha$ is the projection operator on the $\mathrm{H}^\alpha_{\rm ph}$

$$\hat{P}_\alpha = \sum_{n=0}^{\infty} |\Psi_n\rangle\langle\Psi_n| \tag{3.20}$$

and the orthonormal basis Ψ_n in $\mathrm{H}^\alpha_{\rm ph}$ can be chosen as

$$\Psi_n = \exp(-\tfrac{1}{2}zz^*)\frac{z^{*n}}{\sqrt{n!}}. \tag{3.21}$$

Note that the operator $\hat{P}_\alpha$ is self-conjugate and, as a projector on vacuum (with respect to the variables ϕ_p and ϕ_q), can be represented as the normally ordered exponent [3, 11]

$$\hat{P}_\alpha =: \exp(-\hat{\phi}^*_\alpha \hat{\phi}_\alpha) :\equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \hat{\phi}^{*n}_\alpha \hat{\phi}^n_\alpha. \tag{3.22}$$

It is clear that the operators $\hat{R}_{G,ph}$ and $\hat{R}_G$ have the same matrix elements in the $\mathrm{H}^\alpha_{\rm ph}$. Using the representation (3.22), we see that the classical expression of $\hat{R}_{G,ph}$ is contained in the class $\widetilde{R}_G$. Thus, a question arises about the relation between the operators $\hat{R}_{G,ph}$ and the corresponding operators in the holomorphic representation.

For the monomial functions $G = z^n z^{*m}$ (with arbitrary non-negative integer n and m) it can be checked that

$$\hat{R}_{G,ph}\exp(-\tfrac{1}{2}zz^*)\psi(z^*) \\ = \exp(-\tfrac{1}{2}zz^*)\left[\frac{d^n}{dz^{*n}}z^{*m}\psi(z^*) - nm\frac{d^{n-1}}{dz^{*n-1}}z^{*m-1}\psi(z^*)\right].$$

From this, using (3.13)–(3.16), one gets the following correspondence rule

$$\hat{R}_{z^n z^{*m},ph} \to \hat{a}^n\hat{a}^{*m} - nm\hat{a}^{n-1}\hat{a}^{*m-1}$$

with the creation and annihilation operators, $\hat{a}^*$ and $\hat{a}$, from the holomorphic representation.

As a result, for n or $m \leqslant 1$ one obtains normal ordering. For $n > 1$ and $m > 1$ we obtain some unusual ordering.

Therefore, (3.19) is the correct definition of the operators in the physical Hilbert space $\mathrm{H}^\alpha_{\mathrm{ph}}$. It corresponds to the definite operator ordering in the usual canonical quantization.

It can be shown [5] that deforming the operator $\hat{R}_G$ by terms proportional to the first and higher powers of $\hbar$, one can obtain another operator ordering.

3.4. Anomalies

The representation (3.19) is convenient in the analysis of the anomalous commutation relations of the physical operators.

In the physical subspace $\mathrm{H}^\alpha_{\mathrm{ph}}$ the operators (3.19) have the form

$$\hat{R}_{G,ph}|_{\mathrm{H}^\alpha_{\mathrm{ph}}} = \hat{P}_\alpha\hat{R}_G \equiv \hat{R}^\alpha_G.$$

For the commutators one gets

$$\langle\Psi_{\mathrm{ph}}|[\hat{R}^\alpha_G, \hat{R}^\alpha_H]|\Psi_{\mathrm{ph}}\rangle = \langle\Psi_{\mathrm{ph}}|\hat{R}_G\hat{P}_\alpha\hat{R}_H - \hat{R}_H\hat{P}_\alpha\hat{R}_G|\Psi_{\mathrm{ph}}\rangle.$$

Equation (3.23) implies that if one of the operators $\hat{R}_G$ or $\hat{R}_H$ commutes with the projection operator $\hat{P}_\alpha$, then the anomaly in the commutation relations (3.23) is absent.

Inserting the expression (3.22) for the projection operator in (3.23) and using (3.2) we obtain

$$\langle\Psi_{\mathrm{ph}}|[\hat{R}^\alpha_G, \hat{R}^\alpha_H]|\Psi_{\mathrm{ph}}\rangle = -\mathrm{i}\hbar\langle\Psi_{\mathrm{ph}}|\hat{R}^\alpha_{\{G,H\}}|\Psi_{\mathrm{ph}}\rangle + A(G,H)$$

where the anomaly term $A(G,H)$ has the form

$$A = \left(-\langle\hat{R}_G\hat{\phi}^*_\alpha\hat{\phi}_\alpha\hat{R}_H\rangle + \frac{1}{2!}\langle\hat{R}_G\hat{\phi}^*_\alpha\hat{\phi}^*_\alpha\hat{\phi}_\alpha\hat{\phi}_\alpha\hat{R}_H\rangle + \cdots\right) - (G \leftrightarrow H). \tag{3.24}$$

As far as $\hat{\phi}_\alpha|\Psi_{\mathrm{ph}}\rangle = 0$ and $[\hat{R}_H, \hat{\phi}_\alpha] \sim \hbar$ (see (3.7*b*)), we find that the anomaly is quadratic or of higher power in $\hbar$.

For illustration consider the flat phase space with coordinates

$$q^\mu, p_\mu \qquad \mu = 1, 2, \ldots$$

and the 1-form θ

$$\theta = \tfrac{1}{2} p_\mu \mathrm{d}q^\mu - \tfrac{1}{2} q^\mu \mathrm{d}p_\mu .$$

Introducing the complex variables z_k

$$z_k = \frac{\alpha_{|k|} q_{|k|} + \mathrm{i}\varepsilon_k p_{|k|}}{\sqrt{2\alpha_{|k|}}} \quad k = \pm 1, \pm 2, \ldots$$

where

$$\varepsilon_k = \begin{cases} 1 & k > 0 \\ -1 & k < 0 \end{cases} \qquad \alpha_{|k|} > 0$$

for the components of the 1-form θ we have

$$\theta_k = \tfrac{1}{2} \mathrm{i}\varepsilon_k z_{-k} \qquad \text{(without summation)}. \tag{3.25a}$$

Correspondingly, the symplectic matrix ω_{kl} and its inverse take the form (see (2.4))

$$\omega_{kl} = -\mathrm{i}\varepsilon_k \delta_{k+l,0} = (\omega)^{-1}_{kl}. \tag{3.25b}$$

Then, due to (2.7) we obtain the operators $\hat{\phi}_k$

$$\hat{\phi}_k = -\hbar \partial_{z_k} - \tfrac{1}{2} \mathrm{i}\varepsilon_k z_{-k} \tag{3.26a}$$

with commutation relations

$$[\hat{\phi}_k, \hat{\phi}_l] = \hbar \varepsilon_k \delta_{k+l,0}. \tag{3.26b}$$

The vectors of the physical subspace (according to (3.9)) satisfy

$$\hat{\phi}_k |\Psi_{\mathrm{ph}}\rangle = 0 \qquad k > 0. \tag{3.27}$$

Let us now calculate the anomaly $A(G, H)$ for quadratic functions

$$H = \tfrac{1}{2} H_{kl} z_k z_l \qquad G = \tfrac{1}{2} G_{kl} z_k z_l \qquad (H_{kl} = H_{lk} \quad G_{kl} = G_{lk}). \tag{3.28}$$

Using (3.1) and (3.25) we get

$$\hat{R}_H = \hbar \varepsilon_k H_{kj} z_j \partial_{z_{-k}} \qquad \hat{R}_G = \hbar \varepsilon_k G_{kj} z_j \partial_{z_{-k}}. \tag{3.29}$$

In our case, the projection operator has the form

$$\hat{P} =: \exp\left(-\frac{1}{\hbar} \sum_{k>0} \hat{\phi}_{-k} \hat{\phi}_k \right) : .$$

Therefore, for the anomaly term $A(G,H)$ from (3.24) we can write

$$A = \left(-\frac{1}{\hbar}\langle \hat{R}_G \hat{\phi}_{-k} \hat{\phi}_k \hat{R}_H\rangle + \frac{1}{2!\hbar^2}\langle \hat{R}_G \hat{\phi}_{-k} \hat{\phi}_{-l} \hat{\phi}_k \hat{\phi}_l \hat{R}_H\rangle + \cdots\right) - (G \leftrightarrow H). \tag{3.30}$$

Moreover, from (3.26) and (3.29) we have the commutation relations

$$\left[\hat{\phi}_k, \hat{R}_H\right] = \hbar H_{kl} \varepsilon_l \hat{\phi}_{-l}. \tag{3.31}$$

Using (3.26*b*), (3.27) and (3.31) we find that in the expansion (3.30) only the first two terms remain. After the calculation we get

$$A(G,H) = \frac{\hbar^2}{2} \sum_{k,l>0} (G_{kl} H_{-k-l} - G_{-k-l} H_{kl}). \tag{3.32}$$

Finally, observe that (3.32) can be used to compute the conformal anomaly [12]. Indeed, choosing in (3.28) the matrix

$$(H^n)_{kl} = |k|\delta_{k+l,n} \quad k,l = \pm 1, \pm 2, \ldots \qquad n = 0, \pm 1, \pm 2, \ldots$$

we get the generators of conformal transformations [12], and from (3.32) it follows that

$$A^{nm} \equiv A(H^n, H^m) = \delta_{n+m,0} \tfrac{1}{12}(n^3 - n).$$

This is the central extension of Virasoro algebra.

4. Conclusion

In the present work we proposed a new scheme for the quantization of dynamical systems. We have shown that the geometric quantization [2] of Hamiltonian systems may be considered in a certain sense as a covariant quantization of constrained systems. These constrained systems are the extensions of the initial Hamiltonian systems. In addition, the extension scheme is universal and can easily be applied to the case of field theory.

In particular, the proposed quantization scheme provides the possibility to analyse the existence of anomalous terms in the commutation relations of the physical operators.

The structure of the physical operators, (3.19), depends on the choice of the α-parameter. Generally, the operators corresponding to different α's are not unitary equivalent, but for every α they correspond to some operator ordering. In this respect, the dependence of the operator ordering on the existence of anomalies can be studied. We hope that using the representation (3.19) it will be possible to develop a perturbative method for calculating the spectra of physical operators (to find the ground state energy and etc.)

Further development of the above proposed quantization scheme will give a deeper understanding of the connection between the covariant and non-covariant quantizations. We hope to develop this scheme of quantization for a realistic field theory model.

Note added. Since completing this work we became aware of an interesting paper [13] in which a similar analysis of the anomalous terms in the commutation relations (Schwinger terms) was done in the framework of geometric quantization.

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